A re-expansion method for determining the acoustical impedance and the scattering matrix for the waveguide discontinuity problem

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The paper gives a new method for analyzing planar discontinuities in rectangular waveguides. The method consists of a re-expansion of the normal modes in the two ducts at the junction plane into a system of functions accounting for the velocity singularities at the corner points. As the new expansion has an exponential convergence, only a few terms have to be considered for obtaining the solution of most practical problems. To see how the method works some closed form solutions, obtained by the conformal mapping method, are used to discuss the convergence of the re-expanded series when the number of retained terms increases. The equivalent impedance accounting for nonplanar waves into a plane-wave analysis is determined. Finally, the paper yields the scattering matrix which describes the coupling of arbitrary modes at each side of the discontinuity valid in the case of many propagating modes in both parts of the duct.
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I. INTRODUCTION

This work is dedicated to analysis of wave scattering by thin plates and steps in rectangular acoustic waveguides having planar junctions (discontinuities). The main result shows that accurate numerical results can be obtained by using very few terms in the expansion of the particle axial velocity field in the junction plane by using a Galerkin procedure and choosing the approximating functions so that they correctly model the singularities in the velocity field at the discontinuity points in the junction plane of the guide. This way the equivalent impedance of discontinuity, useful in a plane wave-based analysis for small frequencies as well as the scattering matrix of the discontinuity necessary in the case of many propagating modes are given.

For the segments of ducts with constant geometrical and physical properties, in many cases it is possible to have a single modal representation for the solution. When two segments of different properties are connected to each other the modal representation for each of them still holds. But to satisfy the conditions of continuity of pressure and normal velocity at the connection plane (junction) the expansion of the incident field has to be reformulated into an expansion of the transmitted field and also a part of the incident field is reflected. The incident mode is scattered into a modal spectrum of transmitted and reflected modes. The next operation, called mode matching, determines the scattered field by satisfying the continuity conditions. The reflected and transmitted fields contain an infinite number of modes such that the practical mode matching operation involves truncations in the number of modes for each field requiring a certain balance between the accuracy of reflected and transmitted waves.

Miles[1] was among the first to investigate the discontinuity problem for acoustical waves in cylindrical ducts. He wrote the fundamental equations governing the propagation of sound near the discontinuity and applied a mode matching technique to compute plane wave reflection and transmission coefficients. By truncating the equations, Miles obtained approximate solutions for lower frequencies. Karal[2] investigated the acoustic inductance for the sudden discontinuities in two infinite circular ducts of different cross section joined together to form an acoustical transmission system. The given formula for the discontinuity impedance is approximate in the sense that his analysis is valid only at very low frequencies. The problem with these investigations was the arbitrary manner in which the infinite sums in the modal equations were truncated. By using the method of Miles, Alfredson[3] found poor convergence of his solution as the number of considered modes increased. A similar phenomenon was found by the researchers working in electromagnetic waveguide discontinuity problems[4] where different truncated sets give different numerical results.

Kergomard and Garcia[5] used the mode matching method for obtaining the equivalent inductance at planar discontinuities. They have also discussed in more detail the convergence criteria and the number of modes to be considered for various values of the parameters but the work was limited to low frequencies. An analysis of the way different authors used the modal theory in studying the propagation in waveguides with multiple step discontinuities is contained in a paper by Kergomard *et al.*[6]

Powerful mathematical methods have been used for solving the problem of discontinuities in waveguides.

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Rayleigh[7] and Schwinger[8] developed some variational formulations, Lewin[9] considered some integral equations. The book by Mittra and Lee[10] on analytical techniques for guided wave problems contains chapters on Wiener-Hopf techniques and mode-matching methods.

The use of the expansion of the solution in series of approximating functions accounting for the correct modeling of singularities was considered in many physical scattering problems. Amari *et al.*[11] used Chebyshev and Gegenbauer type functions in solving the coupled integral equations for ridged waveguides in some microwave communication systems. Chebyshev polynomials were employed for solving the integral equations of scattering from strips and slots by Tsalamengas *et al.*[12] and Homentcovschi and Miles.[13,14] Bases of weighted Jacobi polynomials satisfying the proper edge condition were used in analyzing circular aperture problems by Grigoryan[15] and Lavretsky.[16] And last but not least, a Galerkin procedure combined with the approximation of the solution in terms of Gegenbauer polynomials was used by Evans and Fernyhough[17] for modeling the edge waves along periodic coastlines. Some of the techniques developed in these papers were used in the present work for analyzing the planar junctions of rectangular waveguides. The "tilde" Gegenbauer polynomials, used for approximation of axial velocity in the aperture, contain as particular cases the Chebyshev polynomials and the classical Gegenbauer polynomials.

As denoted in Solokhin[18] the previous work made significant contributions to the problem but the authors applied some assumptions to simplify the computation. Besides the low frequency hypothesis, the most important geometrical restriction is the symmetry of the composed waveguide. We show that a slight modification of the approach in the case of linear waveguides permits the study of discontinuities in nonsymmetrical cases using basically the same mathematical apparatus as in the case of symmetrical structures.

Two kinds of discontinuities will be studied in this paper: the first is an abrupt change in cross-section (step discontinuity). When an abrupt contraction of the cross section is immediately followed by an expansion we obtain the second type of discontinuity discussed, called diaphragm or iris. The velocity singularities at the edge points on the discontinuity plane involve special phenomena. In order to obtain a unique solution we have to consider the "edge condition" formulated by Mittra and Lee: the energy integral of the total field in a neighborhood of an edge must be finite. On the other hand, the velocity singularities are responsible for the slow convergence of the infinite mode sums. Due to the singularity a lot of modes have to be matched before obtaining an acceptable solution. The situation is similar to the classic Gibbs phenomena when the expansion of a nonsmooth function in a finite Fourier series exhibits spurious oscillations near the region where the function is not smooth. The resolution of this phenomenon will be obtained, using an idea developed by Gottlieb *et al.*,[19] by considering a re-expansion of the velocity in the plane $z=0$ in a set of basic polynomials, orthogonal with a weight function accounting for the singularity of the velocity. The matching of velocity resulting in this expansion with those obtained in modal expansions in the two ducts determine the expansion coefficients in the duct segments and, finally, the continuity of the pressure gives the equation for determining the re-expansion coefficients.

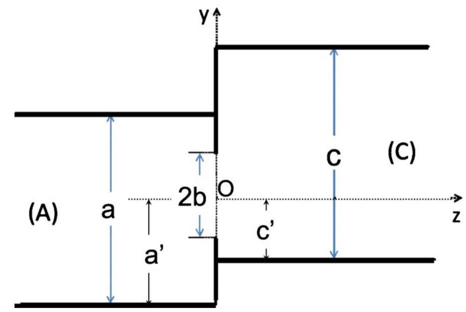

FIG. 1. (Color online) A planar junction between two general rectangular waveguides of equal width.

The method of re-expansion of velocity in the discontinuity plane into special orthogonal polynomials has two important advantages: first, instead of a doubly-infinite set of equations which results by direct matching of modes we have to deal with a single infinite set of equations. The second, and most important benefit, follows as a consequence of considering the explicit form of singularity of velocity in the re-expansion formula: the new form provides an exponential accuracy with a very limited number of terms. Thus, in most practical cases it is sufficient to sum a small number of terms to obtain the physical variables to three significant digits over an entire range of parameters.

II. GENERAL THEORY

A. The basic equations

The analysis will be restricted to a planar junction between two bidimensional rectangular guides (A) and (C) of equal width but unequal heights. In addition, to make the problem more general, a zero thickness obstacle will be inserted in the junction plane as indicated in Fig. 1 (following a structure considered in the symmetrical case by Schwinger *et al.*[8]). The common transverse dimension is denoted by d and the heights of the two coupled rectangular guides are a and c. Finally, the height of the opening (aperture) S_{ap} in the junction plane is denoted by $2b$. A Cartesian coordinate system is introduced in such a way that the Oxy plane coincides with the plane of discontinuity; the origin being placed at the middle of the opening. The z—axis is parallel to the sides of the ducts. The distances of the lower sides of the ducts to the z —*axis* are denoted by a' and c' respectively. Also, we denote by S_A the intersection of the guide (A) with the junction plane $z=0$ and $S_C=(C)\cap\{z=0\}$.

Let $P(\mathbf{x},t)$ denote the pressure and consider the t—periodic solutions

$$P(\mathbf{x},t) = p(\mathbf{x})e^{i\omega t}, \tag{1}$$

where ω is the angular frequency. Then, the complex pressure satisfies Helmholtz's equation

$$\nabla^2 p + k^2 p = 0, \tag{2}$$

where $k=\omega/\tilde{c}$ is the wave number and $\tilde{c}$ denotes the speed of sound.

By assuming rigid duct wall boundary conditions a separation of variables solution can be obtained for the pressure $p(y,z)$ for the two separate ducts, (A) and (C),

$$p_A(y,z) = A_0^r e^{ikz} + A_0^a e^{-ikz} + \sum_{m=1}^{\infty} (A_m^r e^{\gamma_{a,m} z} + A_m^a e^{-\gamma_{a,m} z}) \times \cos\left(\frac{m\pi}{a}(y+a')\right), \quad z<0, \tag{3}$$

$$p_C(y,z) = C_0^r e^{-ikz} + C_0^a e^{ikz} + \sum_{m=1}^{\infty} (C_m^r e^{-\gamma_{c,m} z} + C_m^a e^{\gamma_{c,m} z}) \times \cos\left(\frac{m\pi}{c}(y+c')\right), \quad z>0, \tag{4}$$

where

$$\gamma_{a,m} = \left[\left(\frac{m\pi}{a}\right)^2 - k^2\right]^{1/2} = i\left[k^2 - \left(\frac{m\pi}{a}\right)^2\right]^{1/2} \equiv ik_{a,m}, \tag{5}$$

$$\gamma_{c,m} = \left[\left(\frac{m\pi}{c}\right)^2 - k^2\right]^{1/2} = i\left[k^2 - \left(\frac{m\pi}{c}\right)^2\right]^{1/2} \equiv ik_{c,m}. \tag{6}$$

Also, A_m^a, C_m^a denote the modal amplitudes of the approaching waves and A_m^r, C_m^r represent the receding modal amplitudes in the two connected waveguides. In the case of real $k_{a,m}$ (respectively $k_{c,m}$) the term $\exp[i(\omega t - k_{a,m} z)]$ describes a non-decaying propagating wave (of amplitude A_m^a) in the direction of increasing z while the term $\exp[i(\omega t + k_{c,m} z)]$ gives a wave, of modal amplitude C_m^a, propagating to the left (in the direction of decreasing z). For $m\pi/a > k$ the coefficients $\gamma_{a,m}$ are real and positive and the modal amplitudes A_m^r characterize evanescent modes. Similarly, the modal amplitudes C_m^r correspond in the case $m\pi/c > k$ to evanescent waves. The fundamental difference between the propagating and evanescent waves is that while the propagating modes transport (propagate) the energy the evanescent waves store it locally.

The axial components of the acoustic velocity, v, are given by

$$v_A(y,z) = \frac{-1}{i\omega\rho}\frac{\partial p_A}{\partial z} = \frac{k}{\omega\rho}(A_0^a e^{-ikz} - A_0^r e^{ikz}) + \sum_{m=1}^{\infty} \frac{\gamma_{a,m}}{i\omega\rho}(A_m^a e^{-\gamma_{a,m} z} - A_m^r e^{\gamma_{a,m} z}) \times \cos\left(\frac{m\pi}{a}(y+a')\right), \quad z \le 0, \tag{7}$$

$$v_C(y,z) = \frac{-1}{i\omega\rho}\frac{\partial p_C}{\partial z} = \frac{k}{\omega\rho}(C_0^r e^{-ikz} - C_0^a e^{ikz}) + \sum_{m=1}^{\infty} \frac{\gamma_{c,m}}{i\omega\rho}(C_m^r e^{-\gamma_{c,m} z} - C_m^a e^{\gamma_{c,m} z}) \times \cos\left(\frac{m\pi}{c}(y+c')\right), \quad z \ge 0. \tag{8}$$

Thus, at the plane of discontinuity, $z=0$,we have

$$p_A(y,0-) = \sum_{m=0}^{\infty} (A_m^r + A_m^a)\cos\left(\frac{m\pi}{a}(y+a')\right), \quad y \in S_A, \tag{9}$$

$$p_C(y,0+) = \sum_{m=0}^{\infty} (C_m^r + C_m^a)\cos\left(\frac{m\pi}{c}(y+c')\right), \quad y \in S_C, \tag{10}$$

$$v_A(y,0-) = \sum_{m=0}^{\infty} Y_{a,m}(A_m^a - A_m^r)\cos\left(\frac{m\pi}{a}(y+a')\right), \quad y \in S_A, \tag{11}$$

$$v_C(y,0+) = \sum_{m=0}^{\infty} Y_{c,m}(C_m^r - C_m^a)\cos\left(\frac{m\pi}{c}(y+c')\right), \quad y \in S_C, \tag{12}$$

where

$$Y_{a,m} = \frac{\gamma_{a,m}}{i\omega\rho}, \quad Y_{c,m} = \frac{\gamma_{c,m}}{i\omega\rho}, \quad m \ge 0. \tag{13}$$

The boundary conditions at the discontinuity surface are the following:

$$p_A(y,0-) = p_C(y,0+), \quad y \in S_{ap}, \tag{14}$$

$$v_A(y,0-) = v_C(y,0+), \quad y \in S_{ap}, \tag{15}$$

$$v_A(y,0-) = 0, \quad y \in S_A - S_{ap}, \tag{16}$$

$$v_C(y,0+) = 0, \quad y \in S_C - S_{ap}. \tag{17}$$

Denoting by $\tilde{v}(y)$ the axial velocity in the plane $z=0$ we can write

$$v_A(y,0-) = \begin{cases} \tilde{v}(y), & y \in S_{ap} \\ 0, & y \in S_A - S_{ap}, \end{cases} \tag{18}$$

$$v_C(y,0+) = \begin{cases} \tilde{v}(y), & y \in S_{ap} \\ 0, & y \in S_C - S_{ap}, \end{cases} \tag{19}$$

The formulas (14), (9), and (10) yield the basic equation of the problem

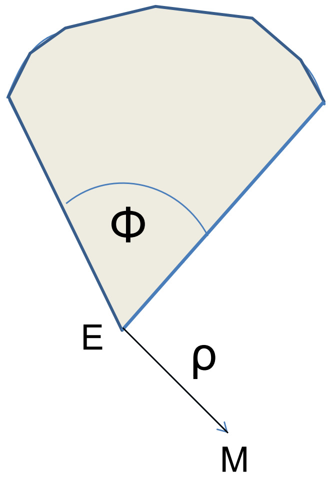

FIG. 2. (Color online) A general reentrant corner.

$$\sum_{m=0}^{\infty}(A_m^r+A_m^a)\cos\left(\frac{m\pi}{a}(y+a')\right)=\sum_{m=0}^{\infty}(C_m^a+C_m^r)\cos\left(\frac{m\pi}{c}(y+c')\right),\quad y\in S_{ap}. \tag{20}$$

B. The velocity singularities at edges

In order to obtain an unique solution of the diffraction problem we have to impose an edge condition stipulating the behavior of velocity at the edge point $(0,\pm b)$. The proper condition formulated by Mittra and Lee[10] requires that the energy integral of the total field in a neighborhood of the edge be finite. At pg.9 it is shown that this condition requires that the physical variable (in our case the velocity v) be of the form

$$v(\rho)=O(\rho^{-1+\tau}),\quad \tau>0, \tag{21}$$

ρ denoting the distance of the current point M to the edge E (see Fig. 2). By assuming the domain as being the exterior of a wedge of ϕ internal angle Collin[20] [Eq. (1.51) pg. 25] has shown that the coefficient τ has the value

$$\tau=\frac{\pi}{2\pi-\phi}. \tag{22}$$

In our analysis we consider two cases:

(1) The problem of a diaphragm of zero thickness (an iris) when the point $(0,b)$ can be considered as the vertex of a wedge of zero internal angle. Therefore, $\phi=0$ and formula (22) gives $\tau=1/2$. In this case the behavior of the velocity in a neighborhood of the edge is given by

$$v(\rho)=O(\rho^{-1/2}). \tag{23}$$

(2) The case of a step discontinuity in a rectangular waveguide when the internal angle of the wedge is $\phi=\pi/2$. Hence $\tau=2/3$ and formula (21) yields

$$v(\rho)=O(\rho^{-1/3}). \tag{24}$$

This is the proper behavior of the velocity at the edge point in the case of a step discontinuity in a duct.

C. The velocity in the aperture

Taking into account the behavior of the velocity field in the aperture at the edge points $y=\pm b$ we write

$$\tilde{v}(y)=\left[1-\left(\frac{y}{b}\right)^2\right]^{\nu-1/2}\sum_{j=0}^{\infty}\tilde{v}_j\tilde{C}_j^{\nu}\left(\frac{y}{b}\right),\quad y\in S_{ap}, \tag{25}$$

$\tilde{C}_j^{\nu}(t)$ being the "tilde" Gegenbauer polynomials defined in the interval $[-1,+1]$ by formulas (A1) and (A2), and $\tilde{v}_j$ the expansion coefficients. Then, Eqs. (11) and (18) yield

$$Y_{a,m}(A_m^a-A_m^r)=\frac{2}{1+\delta_{m,0}}\frac{1}{a}\times\sum_{j=0}^{\infty}\tilde{v}_j\int_{S_{ap}}\left[1-\left(\frac{y}{b}\right)^2\right]^{\nu-1/2}\tilde{C}_j^{\nu}\left(\frac{y}{b}\right)\times\cos\left(\frac{m\pi}{a}(y+a')\right)dy, \tag{26}$$

where $\delta_{0,0}=1$ and $\delta_{m,0}=0$ for $m>0$. By using the formula (A4), the Eq. (26) becomes

$$Y_{a,0}(A_0^a-A_0^r)=\frac{b}{a\Gamma(\nu+1)}v_0, \tag{27}$$

$$Y_{a,m}(A_m^a-A_m^r)=\frac{2}{1+\delta_{m,0}}\frac{b}{a}\sum_{j=0}^{\infty}D_{j,m}^{\nu}\left(\frac{a'}{a},\frac{b}{a}\right)v_j, \tag{28}$$

Here

$$D_{j,m}^{\nu}(u,w)=\cos\left(m\pi u+j\frac{\pi}{2}\right)\frac{J_{j+\nu}(m\pi w)}{[m\pi w/(2)]^{\nu}},\quad j=0,1,2\ldots\quad m=1,2,3,\ldots, \tag{29}$$

$$D_{j,0}^{\nu}(u,w)=\frac{\delta_{j,0}}{\Gamma(\nu+1)},\quad j=0,1,2,\ldots, \tag{30}$$

$$v_j=2^{1-2\nu}\pi\frac{\Gamma(j+2\nu)}{\Gamma(j+1)}\tilde{v}_j. \tag{31}$$

Similarly, Eqs. (12) and (19) give

$$Y_{c,m}(C_m^r-C_m^a)=\frac{2}{1+\delta_{m,0}}\frac{b}{c}\sum_{j=0}^{\infty}D_{j,m}^{\nu}\left(\frac{c'}{c},\frac{b}{c}\right)v_j. \tag{32}$$

To determine the coefficients $\{v_n\}$ we apply a Galerkin procedure by multiplying Eq. (20) by $(1-(y/b)^2)^{\nu-1/2}\tilde{C}_p^{\nu}(y/b)$ and integrating along the interval $(-b,b)$. There results

$$\sum_{m=0}^{\infty}(A_m^r+A_m^a)D_{j,m}^{\nu}\left(\frac{a'}{a},\frac{b}{a}\right)=\sum_{m=0}^{\infty}(C_m^a+C_m^r)D_{j,m}^{\nu}\left(\frac{c'}{c},\frac{b}{c}\right),\quad j=0,1,2,\ldots. \tag{33}$$

III. THE CASE WHEN ONLY THE PLANE MODE PROPAGATES

At the low frequencies (which are of prime importance in silencer design for example) only plane waves can propagate along a uniform duct section and thus plane-wave theory is generally adequate. However, at a duct discontinuity, evanescent non-planar waves are created which cause a noticeable difference between the results from basic plane-wave analysis and the experimental results. The effect of these nonplanar wave modes can be incorporated into a plane-wave analysis by the employment of an equivalent acoustical impedance at the discontinuity introduced by Miles.[1] The purpose of the present section is to determine an expression for the discontinuity impedance which is valid for all frequencies up to the cut-on frequency of non-planar modes with a reduced computational effort. Since in some cases closed form solutions are available (especially for low frequencies) this paper will do a validation of the proposed method by comparing the final results with those given by closed form solutions.

A. Determination of the velocity in aperture

In the case where $ak<\pi$, $ck<\pi$ in both waveguides (A) and (C) only the lowest plane waves propagate. Therefore,

$$A_m^a = C_m^a = 0, \quad m = 1,2,3, \ldots ,$$

and we also write

$$A_0^r + A_0^a = A_0, \quad C_0^r + C_0^a = C_0,$$

assuming general termination conditions different from the anechoic termination of duct (C) supposed by Karal.[2] The relationship (33) becomes

$$\sum_{m=1}^{\infty} C_m^r D_{j,m}^{\nu}\left(\frac{c'}{c},\frac{b}{c}\right) - \sum_{m=1}^{\infty} A_m^r D_{j,m}^{\nu}\left(\frac{a'}{a},\frac{b}{a}\right) = \frac{A_0 - C_0}{\Gamma(\nu+1)}\delta_{j,0}, \quad j = 0,1,2, \ldots . \tag{34}$$

Substituting the coefficients A_m, C_m by using the formulas (28) and (32) we can write the relationships

$$\sum_{j=0}^{\infty} [\Delta_{p,j}^{(\nu)}(a,a') + \Delta_{p,j}^{(\nu)}(c,c')]v_j^0 = \delta_{p,0}, \quad p = 0,1,2, \ldots , \tag{35}$$

where it has been denoted

$$\Delta_{p,j}^{(\nu)}(a,a') = \sum_{n=1}^{\infty} \cos\left(n\pi\frac{a'}{a} + p\frac{\pi}{2}\right)\cos\left(n\pi\frac{a'}{a} + j\frac{\pi}{2}\right) \times \frac{2J_{p+\nu}(n\pi b/a)J_{j+\nu}(n\pi b/a)}{\pi\sqrt{n^2 - (ak/\pi)^2}[n\pi b/(2a)]^{2\nu}}, \tag{36}$$

and the new unknown coefficients $\{v_j^0\}$ are defined by

$$v_j^0 = \frac{i\omega\rho b}{C_0 - A_0}\Gamma(\nu+1)v_j. \tag{37}$$

The relationships (35) provide an infinite system of linear equations for determining the velocity coefficients $v_j^0 (j=0,1,2,\ldots)$.

Remark 1. In the case $\nu=0$ we have

$$v_0^0 = \frac{i\omega\rho\pi b}{C_0 - A_0}\tilde{v}_0.$$

B. Calculation of the discontinuity impedance

We define, following Karal,[2] the volume velocity

$$V = d\int_{-a'}^{a-a'} v_A(y,-0)dy = d\int_{-c'}^{c-c'} v_C(y,+0)dy, \tag{38}$$

d being the x–dimension common to both guides. Therefore,

$$V = adY_{a,0}(A_0^a - A_0^r) = cdY_{c,0}(C_0^r - C_0^a). \tag{39}$$

Equation (34) yields for $j=0$:

$$A_0 - C_0 = \Gamma(\nu+1)\left(\sum_{m=1}^{\infty} C_m^r D_{0,m}^{\nu}\left(\frac{c'}{c},\frac{b}{c}\right) - \sum_{m=1}^{\infty} A_m^r D_{0,m}^{\nu}\left(\frac{a'}{a},\frac{b}{a}\right)\right). \tag{40}$$

Here $A_0=(p_{0A})_{z=0}$ is the zeroth mode pressure in the domain (A) and $C_0=(p_{0C})_{z=0}$ is the corresponding pressure mode in the region (C).The Eq. (40) indicates that the pressure in the zeroth order mode in the two regions is discontinuous across the section $z=0$. The discontinuity of pressure will be represented by a lumped impedance Z at the change of cross section

$$(p_{0A})_{z=0} - (p_{0C})_{z=0} = VZ^{(\nu)}. \tag{41}$$

The formulas (39) and (41) give the expression for the impedance

$$Z^{(\nu)}d = \frac{A_0 - C_0}{aY_{a,0}(A_0^a - A_0^r)}.$$

By introducing the expression for $Y_{a,0}(A_0^a - A_0^r)$ given by formula (27) and taking into account the formula (37) there results

$$Z^{(\nu)}d/\rho = -i\omega\frac{\Gamma^2(\nu+1)}{v_0^0}.$$

Hence,

$$L^{(\nu)}d/\rho = \frac{\Gamma^2(\nu+1)}{v_0^0}, \tag{42}$$

where $L^{(\nu)}$ is the analogous *acoustical inductance* of the plane discontinuity.

C. The case of a symmetrical geometry

In the case of a symmetrical geometry the problem can be studied for the structure in Fig. 3 which is a half of the

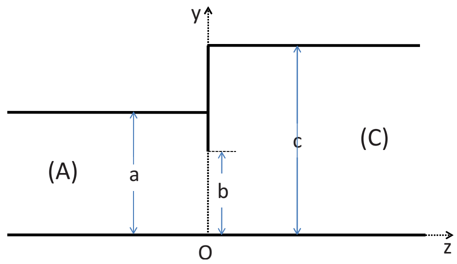

FIG. 3. (Color online) The junction between two symmetrical waveguides.

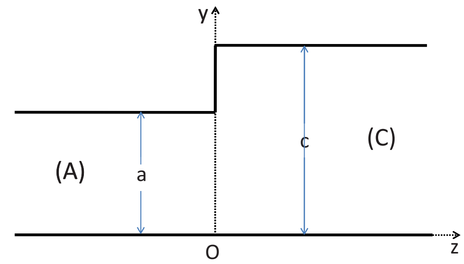

FIG. 4. (Color online) A step discontinuity in a rectangular waveguide (symmetrical case).

whole domain. In fact in many applications the very domain in Fig. 3 is important. This is why we give here also the formulas corresponding to the symmetrical case.

First of all due to symmetry we have $v^0_{2j'+1}=0$. The system (35) becomes

$$\sum_{j'=0}^{\infty}[\Delta^{(\nu)}_{2p',2j'}(a)+\Delta^{(\nu)}_{2p',2j'}(c)]v^0_{2j'}=\delta_{2p',0},\quad p'=0,1,2,\ldots, \tag{43}$$

where the coefficients can be written as

$$\Delta^{(\nu)}_{2p',2j'}(a)=\sum_{n=1}^{\infty}\frac{J_{2p'+\nu}(n\pi b/a)J_{2j'+\nu}(n\pi b/a)}{\pi\sqrt{n^2-(ak/\pi)^2}[n\pi b/(2a)]^{2\nu}}. \tag{44}$$

The analogous acoustical inductance of the structure discontinuity is again determined by formula (42).

D. Examples

In order to see how the re-expansion method works we have computed the inductance of the change in the cross-section for the case of a step discontinuity and also for the case of a diaphragm in a rectangular pipe of constant section.

As compared with the classical modal theory a first advantage is that instead of solving a double infinite system of linear equations (involving an infinite number of modes on each side of the discontinuity) we have to deal with a simple infinite system of equations determining the expansion coefficients of axial velocity in the discontinuity plane. This way, the tedious discussion about the number of modes which have to be summed in each side of the discontinuity is completely avoided. For analyzing how efficient the proposed re-expansion method is we will apply this solution technique to the case of very low frequencies where, in the plane wave case, some analytical solutions obtained by the conformal mapping method are available.

1. The step discontinuity in a rectangular wave guide

For the case of the waveguide step discontinuity $a=b<c$ in Fig. 4 the discontinuity impedance determined by the conformal mapping method can be found in the book by Morse & Ingard, Ref. 21 formula (9.1.28) pg. 488

$$\frac{Ld}{\rho}=\frac{1}{\pi}\left[\frac{(c-a)^2}{2ac}\ln\frac{c+a}{c-a}+\ln\frac{(c+a)^2}{4ac}\right]. \tag{45}$$

The same discontinuity impedance, determined by using the re-expansion technique, is given by formula (26) where the coefficient v_0 results by solving the infinite system of Eqs. (43). In this case we have $\nu=1/6$. By N is denoted the number of equations (and also the number of unknown coefficients) retained in the infinite system after truncation. In Fig. 5 is plotted the variation of the coefficient Ld/ρ with the ratio b/c: the coefficient resulting from the conformal method [formula (45)] is shown by a continuous line, the coefficient resulting from formula (26) when in the system (43) is considered a single equation ($N=1$) is shown by the broken line. The coefficient resulting when $N=2$ is shown as full triangles (▲) and, finally, the case $N=4$ is plotted as full circles (●). For small values of the ratio b/c all the coefficients resulting by using formula (26) give good approximations of the discontinuity impedance. The approximation improves as the number of equations retained in the infinite system increases. For the case $N=4$, corresponding to truncation of the system to the first four equations, (circles in Fig. 5) the computed coefficient practically coincides with that given by analytical formula (45).

Since in this case the exact solution is known we can evaluate the relative error of the approximation of the discontinuity impedance obtained by solving the truncated system (43) and formula (26). The curves corresponding to the

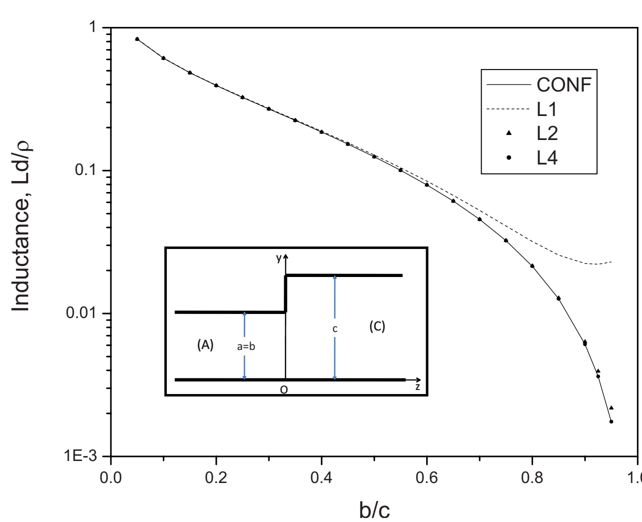

FIG. 5. (Color online) Variation of the inductance of a step discontinuity with the ratio b/c.

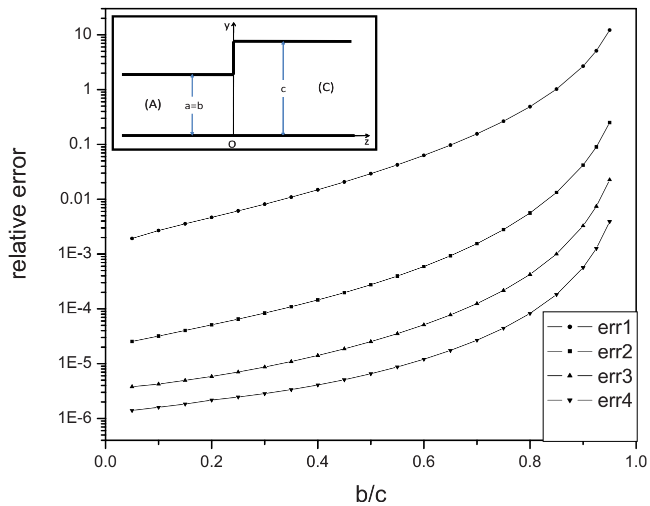

FIG. 6. (Color online) The plot of the relative errors corresponding to the cases where 1, 2, 3, and 4 equations were retained in the infinite set of equations.

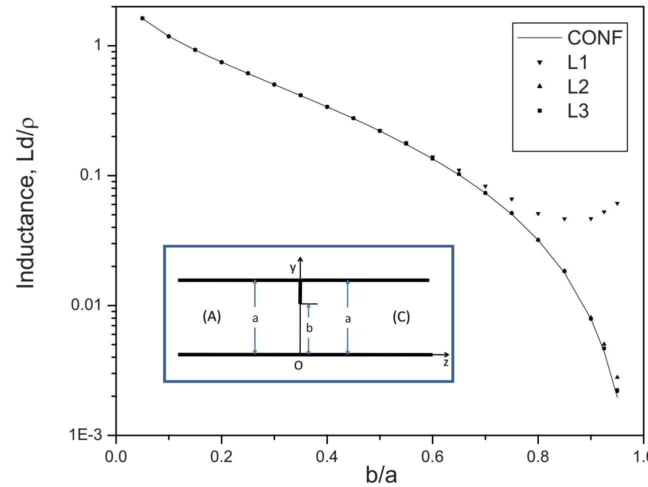

FIG. 7. (Color online) The discontinuity inductance in the case of a diaphragm.

case where we consider a single equation ($N=1$), two equations ($N=2$), three equations ($N=3$) and, finally, four equations ($N=4$) in the system (43) are shown in Fig. 6. The plots indicate the exponential convergence of the approximate solution to the exact discontinuity impedance.

2. The diaphragm in a rectangular guide of constant section

For a diaphragm in a plane rectangular guide the static (low frequency) impedance can also be obtained as a closed formula by conformal mapping. Thus, in the same book by Morse and Ingard[21] is given the formula (9.1.26)

$$\frac{Ld}{\rho} = -\frac{2}{\pi}\log\left[\sin\left(\frac{\pi b}{2a}\right)\right]. \tag{46}$$

The discontinuity impedance resulting by using the re-expansion method is given by the general formula (26) where the coefficient $\tilde{v}_0$ is obtained by solving the infinite system of Eqs. (35). As was stated before, in this case we have $\nu=0$.The results obtained by truncating the infinite system to one equation (▼), to two equations (▲) and three equations (■) are shown in Fig. 7 together with the exact value obtained by conformal mapping (continuous line). The conclusions for the case of the step discontinuity continue to subsist for the case of the diaphragm. The solution can be obtained with a very limited number of terms for most practical cases.

IV. THE SCATTERING MATRIX OF A DISCONTINUITY

In this section the solution of the planar junction problem is given in the form of a scattering matrix which describes the coupling of arbitrary modes at each side of the discontinuity. The scattering matrix represents the discontinuity untied from the influence of sound sources and reflecting terminations of the waveguide. The solution remains valid at high frequencies when many propagating modes exist in both parts of the waveguide. Having determined the scattering matrix, arbitrary arrangements with different sources and reflection coefficients of each higher mode can easily be calculated.

By introducing the matrices

$$[Y_a] = \mathrm{diag}\left(\sqrt{\frac{(1+\delta_{m,0})a}{2b}}Y_{a,m}\right), \quad m=0,1,2,\ldots,M, \tag{47}$$

$$[Y_c] = \mathrm{diag}\left(\sqrt{\frac{(1+\delta_{m,0})c}{2b}}Y_{c,m}\right), \quad m=0,1,2,\ldots,M, \tag{48}$$

$$[D_a] = \left[D_{j,m}\left(\frac{a'}{a},\frac{b}{a}\right)\right]_{j,m=0,1,2,\ldots,M}, \tag{49}$$

and the vectors

$$\mathbf{a}^a = [A_0^a, A_1^a, A_2^a, \ldots, A_M^a]^T, \quad \mathbf{a}^r = [A_0^r, A_1^r, A_2^r, \ldots, A_M^r]^T,$$

$$\mathbf{c}^a = [C_0^a, C_1^a, C_2^a, \ldots, C_M^a]^T, \quad \mathbf{c}^r = [C_0^r, C_1^r, C_2^r, \ldots, C_M^r]^T$$

$$\mathbf{v} = [v_0, v_1, v_2, \ldots v_M]^T,$$

the relationships (28), (32), and (33) become

$$[Y_a](\mathbf{a}^a - \mathbf{a}^r) = [D_a]^T\mathbf{v}, \tag{50}$$

$$[Y_c](\mathbf{c}^r - \mathbf{c}^a) = [D_c]^T\mathbf{v}, \tag{51}$$

$$[D_a](\mathbf{a}^a + \mathbf{a}^r) = [D_c](\mathbf{c}^a + \mathbf{c}^r). \tag{52}$$

A. The scattering matrix

The reflection and transmission coefficients are defined by scattering parameters which relate the receding modal amplitudes by the approaching modal amplitudes Hude *et al.*,[22] Muehleisen *et al.*[23]

$$A_m^r = \sum_n S_{m,n}^{11} A_n^a + \sum_n S_{m,n}^{12} C_n^a, \tag{53}$$

$$C_m^r = \sum_n S_{m,n}^{21} A_n^a + \sum_n S_{m,n}^{22} C_n^a. \quad (54)$$

In these relationships, the coefficient $S_{m,n}^{11}$ represents the reflection coefficient in waveguide (A) from mode n to mode m, $S_{m,n}^{22}$ is the reflection coefficient in waveguide (C) from mode n to mode m. Also, $S_{m,n}^{12}$ is the transmission coefficient from mode n in waveguide (C) to mode m in waveguide (A). Finally, $S_{m,n}^{21}$ is the transmission coefficient from mode n in waveguide (A) to mode m in waveguide (C). The plane wave reflection coefficients are $S_{0,0}^{11}$ and $S_{0,0}^{22}$ and the plane transmission coefficients are $S_{0,0}^{12}$ and $S_{0,0}^{21}$. By using the previously introduced notations, formulas (53) and (54) become

$$\mathbf{a}^r = [S^{11}]\mathbf{a}^a + [S^{12}]\mathbf{c}^a, \quad (55)$$

$$\mathbf{c}^r = [S^{21}]\mathbf{a}^a + [S^{22}]\mathbf{c}^a. \quad (56)$$

These relationships hold independent of the values of the amplitudes of approaching modes $\mathbf{a}^a, \mathbf{c}^a$. Particularly, by taking $\mathbf{c}^a = 0$ formulas (50) and (51) yield

$$\mathbf{v} = [D_a^T]^{-1}[Y_a][I - S^{11}]\mathbf{a}^a, \quad (57)$$

$$\mathbf{v} = [D_c^T]^{-1}[Y_c][S^{21}]\mathbf{a}^a, \quad (58)$$

and hence

$$[S^{21}] = [Y_c]^{-1}[D_c^T][D_a^T]^{-1}[Y_a][I - S^{11}]. \quad (59)$$

Also, the relationship (52) gives

$$[D_a][I + S^{11}] = [D_c][S^{21}], \quad (60)$$

and formula (59) becomes

$$[D_a][I + S^{11}] = [H_a][I - S^{11}], \quad (61)$$

where

$$[H_a] = [D_c][Y_c]^{-1}[\ D_c^T)][D_a^T]^{-1}[Y_a]. \quad (62)$$

Finally, we obtain

$$[S^{11}] = [H_a + D_a]^{-1}[H_a - D_a], \quad (63)$$

$$[S^{21}] = 2[D_c]^{-1}[D_a][H_a + D_a]^{-1}[H_a]. \quad (64)$$

Similarly, considering $\mathbf{a}^a = 0$, we obtain

$$[S^{22}] = [H_c + D_c]^{-1}[H_c - D_c], \quad (65)$$

$$[S^{12}] = 2[D_a]^{-1}[D_c][H_c + D_c]^{-1}[H_c], \quad (66)$$

where

$$[H_c] = [D_a][Y_a]^{-1}[D_a^T][D_c^T]^{-1}[Y_c]. \quad (67)$$

The formulas (63)–(66) permit the complete determination of the scattering matrix of the discontinuity.

B. Case study: A symmetric junction

We analyze the coupling coefficients for a symmetric junction of two rectangular ducts of equal height in Fig. 8 and calculate the reflection and transmission coefficients. This type of junction is regularly encountered in active noise control at higher frequencies. It is important to know how much of a plane wave is converted into higher order modes

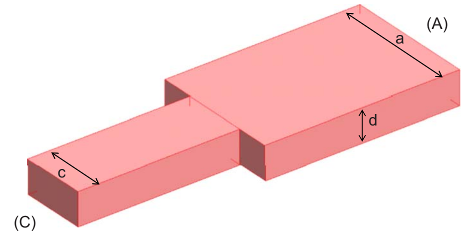

FIG. 8. (Color online) A planar junction of two rectangular waveguides of equal height.

at a discontinuity and also how to design a series of planar discontinuities to convert plane waves to higher order modes where the energy can be more efficiently dissipated.[24]

In this case there is no discontinuity in the x direction so there is no scattering in this direction and the x-dependence in the problem drops out. In Fig. 9 the magnitude and phase of the plane wave reflection coefficient of the larger waveguide (A), S_{00}^{11}, for three different ratios $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$ are presented as functions of ka/π. At low

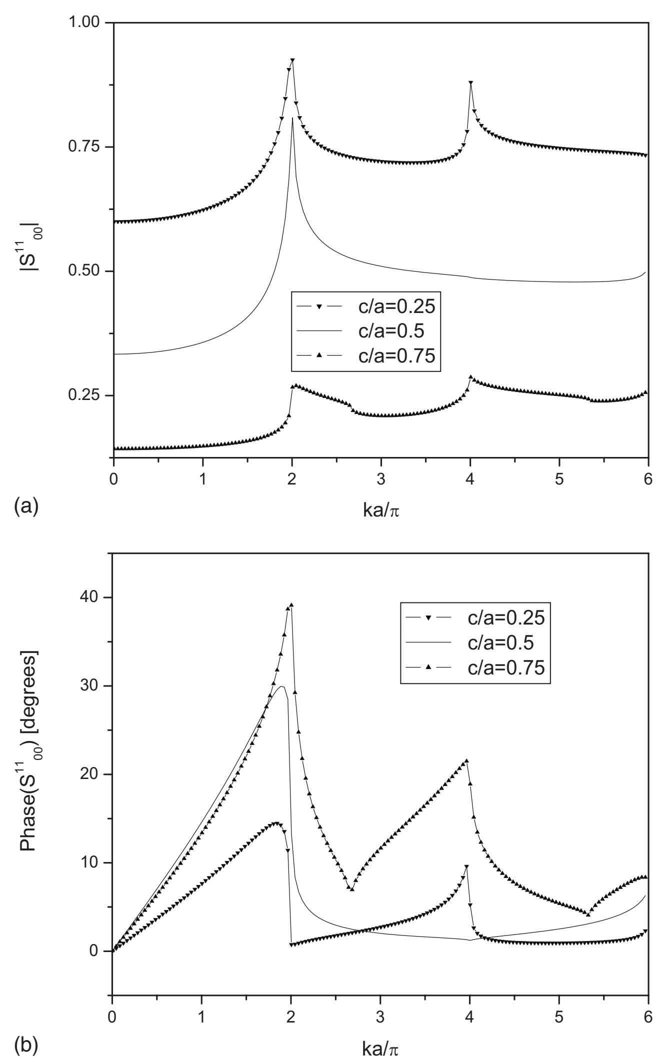

FIG. 9. (a) The magnitude of the plane wave reflection coefficient S_{00}^{11} for the symmetric junction in Fig. 8 with ratio of widths $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$. (b) The phase of the plane wave reflection coefficient S_{00}^{11} for the symmetric junction in Fig. 8 with ratio of widths $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$.

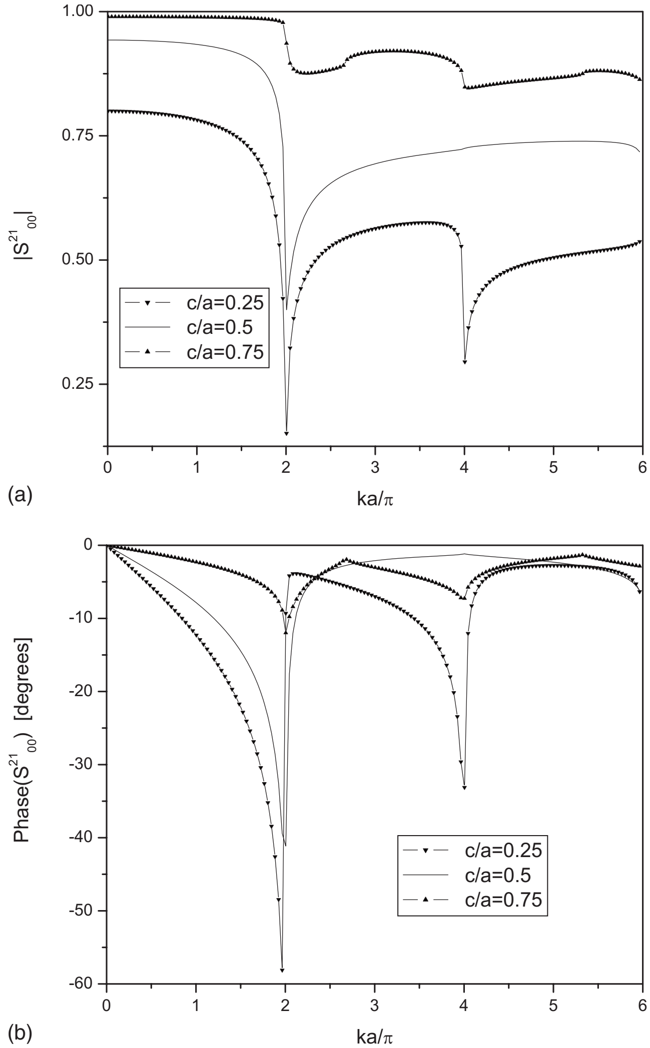

FIG. 10. (a) The magnitude of the plane wave transmission coefficient S_{00}^{21} for the symmetric junction in Fig. 8 with ratio of widths $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$. (b) The phase of the plane wave transmission coefficient S_{00}^{21} for the symmetric junction in Fig. 8 with ratio of widths $c/a=0.25$, $c/a=0.5$, and $c/a=0.75$.

frequencies S_{00}^{11} is well approximated by the plane wave limit of $(a-c)/(a+c)$. As the frequency approaches the cutoff frequency of the first higher order mode in the waveguide (A) at $ka/\pi=2$, the evanescent higher modes begin to store part of the energy from the incident plane wave which can be seen in the increase in the magnitude and phase of S_{00}^{11}. Above the cutoff frequency of the first higher order mode, the higher order mode admittance becomes real, creating another propagating outlet for the incident energy. This leads to a sudden drop in magnitude and phase of the reflection coefficient. This scenario repeats at increasingly higher modes in the larger waveguide (A).

Figure 10 shows the magnitude and the phase of the plane wave transmission coefficient from the large waveguide (A) to the small waveguide (C). The magnitude starts at the low frequency value $2\sqrt{ac}/(a+c)$, at first drops slowly and then more quickly as the frequency approaches the cutoff frequency of the first higher order mode. Above this cutoff frequency there is a sharp rise in the magnitude of transmission coefficient. Near the cutoff frequency of the other higher order modes of the larger waveguide (A) there are additional dips in transmission coefficient.

V. CONCLUSION

The problem of diffraction of a plane acoustic wave propagating in a rectangular wave guide by a step discontinuity in the cross section or a diaphragm has been approached by the method of mode matching. This method requires the solution of large, poorly conditioned systems of linear equations obtained by truncation of slowly convergent infinite systems. The slow convergence of the method originates from the singularity of fluid particle velocity at corner points of the domain.

In this paper the velocity in the aperture is represented by special orthogonal base functions accounting for the velocity singularities at the reentrant corner points. The normal-mode representation of the pressure and velocity are re-expanded in this new basis and the continuity conditions at the aperture yield a new infinite system of equations for the unknown coefficients of the basis functions. The proper accounting of the velocity singularities at the aperture edges gives rapidly converging systems of equations.

The new technique is applied to the general case of two rectangular waveguides having a planar junction. This enables the analysis of junctions involving step discontinuities and diaphragms as well. In the case of structures which can be extended to symmetrical ones, special formulas are provided. To examine the effectiveness of the re-expansion method it is applied to some particular structures which have quasi-static closed analytical formulas obtained by conformal mappings. For both cases, a step discontinuity and an unsymmetrical diaphragm, the results show that the solution of most practical problems can be obtained with a very limited number of terms.

The reexpansion method was also applied to determine the scattering matrix of a discontinuity. This matrix permits determination of coupling of arbitrarily many modes at each side of the discontinuity, remaining also valid at high frequencies when there are many propagating modes in both parts of the duct.

The method is very suitable for the case of multiple discontinuities in waveguides where the number of resulting equations for analyzing the structure by the mode matching method are prohibitively large.

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APPENDIX: THE “TILDE” GEGENBAUER POLYNOMIALS $\tilde{C}_n^{\nu}(x)$

The “tilde” Gegenbauer polynomials are defined for $\nu > 0$ as

$$\tilde{C}_n^\nu(x) = \Gamma(\nu) C_n^\nu(x) \equiv \sum_{m=0}^{[n/2]} \frac{(-1)^m \Gamma(\nu+n-m)}{m!(n-2m)!}(2x)^{n-2m}, \tag{A1}$$

and

$$\tilde{C}_n^0(x) = \begin{cases} 2T_n(x)/n, & n \geq 1, \\ 1, & n = 0. \end{cases} \tag{A2}$$

Here $C_n^\nu(x)$ is the Gegenbauer's ultraspherical polynomial and $T_n(x)$ denotes the Chebyshev's polynomial. In the definition (A2) the relationship

$$\lim_{\nu \to 0} \tilde{C}_n^\nu(x) = \frac{2}{n} T_n(x),$$

proved by Erde[25] pg. 177, has been used.

The "tilde" Gegenbauer polynomials satisfy the relation

$$\tilde{C}_n^\nu(-x) = (-1)^n \tilde{C}_n^\nu(x) \tag{A3}$$

and also the orthogonality relationships

$$\int_{-1}^{+1} \tilde{C}_m^\nu(t)\tilde{C}_n^\nu(t)[1-t^2]^{\nu-1/2} dt = \frac{\pi 2^{1-2\nu}\Gamma(n+2\nu)}{n!(n+\nu)} \delta_{m,n}, \tag{A4}$$

$$\int_{-1}^{+1} [\tilde{C}_m^0(t)]^2 \frac{dt}{(1-t^2)^{1/2}} = \pi. \tag{A5}$$

The formula

$$\exp\{izx\} = \sum_{n=0}^{\infty} \tilde{f}_n^\nu \tilde{C}_n^\nu(x), \tag{A6}$$

where

$$\tilde{f}_n^\nu = (i)^n (n+\nu) J_{n+\nu}(z) \left(\frac{2}{z}\right)^\nu, \quad \nu > 0,$$

$$\tilde{f}_n^0 = J_0(z),$$

can be obtained by using a formula given in the book by Boyd[26] on page 502. Now, by multiplying the relationship (73) by $(1-t^2)\tilde{C}_m^\nu(x)$ and integrating over the interval $[-1,1]$ we obtain

$$\int_{-1}^{+1} [1-t^2]^{\nu-1/2} \tilde{C}_m^\nu(t) e^{izt} dt = 2^{1-2\nu} \pi i^m \frac{\Gamma(m+2\nu) J_{\nu+m}(z)}{m!(z/2)^\nu}. \tag{A7}$$

This relationship can be obtained also by using some formulas given by Erde[25] (pg. 38 and 94) and Evans and Fernyhough.[17]

In the limit case $\nu=0$ the formulas (A4) and (A7) become

$$\int_{-1}^{+1} \frac{\tilde{C}_m^0(t)\tilde{C}_n^0(t)}{\sqrt{1-t^2}} dt = \frac{2\pi}{n^2} \delta_{mn}, \tag{A8}$$

$$\int_{-1}^{+1} \frac{\tilde{C}_{2n}^0(t)}{\sqrt{1-t^2}} \cos(zt) dt = \frac{(-1)^n \pi}{n} J_{2n}(z), \tag{A9}$$

$$\int_{-1}^{+1} \frac{\tilde{C}_0^0(t)}{\sqrt{1-t^2}} \cos(zt) dt = \pi J_0(z). \tag{A10}$$

Also, for $\nu=1/6$ we obtain

$$\int_{-1}^{+1} (1-t^2)^{-1/3} \tilde{C}_m^{1/6}(t) \tilde{C}_n^{1/6}(t) dt = \frac{\pi 2^{2/3} \Gamma(n+1/3)}{n!(n+1/6)} \delta_{m,n} \tag{A11}$$

and

$$\int_{-1}^{+1} \frac{\tilde{C}_{2m}^{1/6}(t)\cos(zt)}{(1-t^2)^{1/3}} dt = 2^{2/3}(-1)^m \pi \frac{\Gamma(2m+1/3) J_{2m+1/6}(z)}{(2m)!(z/2)^{1/6}}. \tag{A12}$$

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